Generalized C^1 quadratic B-splines generated by Merrien subdivision algorithm and some applications.

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Abstract. A new global basis of B-splines is defined in the space of generalized quadratic splines (GQS) generated by Merrien subdivision algorithm. Then, refinement equations for these B-splines and the associated corner-cutting algorithm are given. Afterwards, several applications are presented. First a global construction of monotonic and/or convex generalized splines interpolating monotonic and/or convex data. Second, convergence of sequences of control polygons to the graph of a GQS. Finally, a Lagrange interpolant and a quasi-interpolant which are exact on the space of affine polynomials and whose infinite norms are uniformly bounded independently of the partition.

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§1. Introduction and notations

This paper is a continuation of [7] and [10] where a one-parameter family of C^1 Hermite interpolants was defined by the Merrien subdivision algorithm ([6], abbr. MSA). This family provides a general solution to arbitrary monotone and convex Hermite interpolation problems with data at the ends of a single interval. Here we extend and complete these results by considering Hermite interpolation on a partition of a given interval by C^1 functions whose restrictions to subintervals are generated by MSA. Let $X := \{a = x_0, x_1, \ldots, x_n = b\}$ be such a partition of $I = [a, b]$ with $I_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$ for $1 \le i \le n$. Let $\beta = (\beta_1, \ldots \beta_n)$ be a sequence of parameters $\beta_i \in [-1, 0[$.
On each subinterval I_i, we consider the 4-dimensional space $V(\beta_i)$ generated by Merrien subdivision algorithm and depending on the specific pair of parameters (α_i, β_i), where $\alpha_i = \frac{\beta_i}{4(1-\beta_i)} \in [-\frac{1}{8}, 0[$. In that case, it was proved that the MSA is C^1-convergent. Moreover, for $f \in V(\beta_i)$, f' is Hölder and satisfies the following inequality for some constant $C > 0$ and for $\gamma_i = -log_2(1 + \frac{1}{2}\beta_i)$ (see [6], proposition 2):

$$|f'(x) - f'(y)| \le C|x - y|^{\gamma_i}, \ \ (x, y) \in I_i \times I_i.$$

We denote by $GS_2(I, X, \beta)$ or simply $GS_2(\beta)$ the subspace of all functions $g \in C^1(I)$ whose restriction to I_i is in $V(\beta_i)$ for $1 \leq i \leq n$. The dimension of this space is $2n+2$ since g is uniquely determined by Hermite data $y_i = g(x_i)$ and $p_i = g'(x_i)$ for $0 \leq i \leq n$. The elements of $GS_2(\beta)$ are called *generalized quadratic splines* (abbr. GQS) because their properties are similar to those of classical quadratic splines, which correspond to the choice $\beta_i = -1, 1 \leq i \leq n$.
Here is an outline of the paper. In section 2, we define a *new global basis of B-splines* for the space $GS_2(\beta)$ of GQS. In section 3, we use these B-splines to express in a new global form the results on *monotonicity and convexity preserving properties* of GQS, already given in [7][10] for functions defined on one subinterval. In section 4, we also give a global version of the algorithms constructing monotonic or convex interpolants given in the same papers. When compared to other C^1 Hermite interpolants of the literature, the advantage of our method is its simplicity both in the local construction of the GQS and in the adaptivity of the method to arbitrary sets of data. For example, in [1][2], the method is adaptive, however it may need polynomials of arbitrary high degree, which can lead to rather complicated expansions and calculations.
In section 5, we give the *refinement equation* for coarse B-splines in terms of fine B-splines in the space of GQS defined on a refinement of the initial partition. From this result we deduce a *corner-cutting algorithm* (see [9] for definitions and properties) which is the geometric form of the algorithm expressing the new coefficients of a GQS g in the B-spline basis of the fine space in terms of its old coefficients in the B-spline basis of the coarse space. We also prove the convergence to the graph of g of the sequence of control polygons associated with successive steps of this algorithm.
Finally, in section 6, we study two approximation operators: a *Lagrange interpolant* and a *quasi-interpolant* which are both exact on the space $\mathbb{P}_1$ of affine polynomials and whose uniform norm are *uniformly bounded independently of the given partition* on I. This extends a previous result given for ordinary quadratic splines by Kammerer, Reddien and Varga [4] and also by Marsden [5]. We postpone numerical applications to a further paper which should contain variants and refined versions of the general algorithms presented in Section 4 of the present paper.

For the sake of clarity, we now recall the basic equations of the MSA giving the values at the midpoint $m = \frac{1}{2}(a+b)$ of $[a, b]$ of a function f and its first derivative f' from the four values:

$$\{f(a), f'(a); f(b), f'(b)\}$$

at the endpoints of the interval (see [5]). The construction starts with $[a, b] = I_i = [x_{i-1}, x_i]$ and gives the values of f and f' at the dyadic points of I_i. Let $h = b - a$ and $\theta_i = \frac{1}{2}\frac{\beta_i}{\beta_i - 1} = -2\alpha_i \in]0, \frac{1}{4}]$, then

$$f(m) = \frac{1}{2}\left((f(a) + f(b)) - \theta_i h(f'(b) - f'(a))\right)$$

$$f'(m) = \frac{1}{1 - 2\theta_i}\left(\frac{f(b) - f(a)}{h} - 2\theta_i \frac{f'(b) + f'(a)}{2}\right)$$

In each subinterval I_i, let us define the two points

$$\xi_i = x_i - \theta_i h_i, \quad \eta_i = x_i + \theta_{i+1} h_{i+1},$$

with $\xi_0 = x_0$ and $\eta_n = x_n$. Then each element $g_i \in V(\beta_i)$ can be expressed as

$$g_i = a_{i-1} b_0 + d_{i-1} b_1 + c_i b_3 + a_i b_4$$

in the *local B-spline basis* $\{b_0, b_1, b_2, b_3\}$ of $V(\beta_i)$ defined in [10]. By definition, the quadruplet $[a_{i-1}, d_{i-1}, c_i, a_i]$ is the list of *B-coefficients* of g_i on the subinterval I_i. The *local control polygon* (abbr. LCP) of g_i has the four following *local control vertices*

$$\tilde{a}_{i-1} = (x_{i-1}, a_{i-1}), \quad \tilde{d}_{i-1} = (\eta_{i-1}, d_{i-1}), \quad \tilde{c}_i = (\xi_i, c_i), \quad \tilde{a}_i = (x_i, a_i).$$

The ordinates of these points are the B-coefficients of g_i and they can be expressed in function of the four *Hermite data* $(y_{i-1}, p_{i-1}; y_i, p_i)$ at the two end-points of I_i:

$$y_i = g_i(x_i) = g_{i+1}(x_i), \quad p_i = g_i'(x_i) = g_{i+1}'(x_i).$$

They are given, for $0 \le i \le n-1$, by

$$a_i = y_i, \quad c_i = a_i - \theta_i h_i p_i, \quad d_i = a_i + \theta_{i+1} h_{i+1} p_i, \quad a_{i+1} = y_{i+1}$$

(with the convention $h_0 = h_{n+1} = 0$).
Using the MSA and the properties $\alpha_i = -\frac{1}{2}\theta_i$, $\beta_i = \frac{2\theta_i}{2\theta_i - 1}$, $1 - \beta_i = \frac{1}{1-2\theta_i}$, and $\xi_i - \eta_{i-1} = (1 - 2\theta_i)h_i$, we obtain respectively at the midpoint $m_i = \dfrac{x_{i-1} + x_i}{2}$ of I_i:

$$g_i(m_i) = \frac{1}{2}\left((a_{i-1} + a_i) - \theta_i h_i (p_i - p_{i-1})\right) = \frac{1}{2}(d_{i-1} + c_i),$$
$$g_i'(m_i) = \frac{1}{1 - 2\theta_i}\left(\frac{a_i - a_{i-1}}{h_i} - 2\theta_i \frac{p_{i-1} + p_i}{2}\right) = \frac{c_i - d_{i-1}}{\xi_i - \eta_{i-1}},$$

which proves that the tangent to the curve at m_i is the segment $\tilde{d}_{i-1}\tilde{c}_i$.
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§2. Generalized quadratic B-splines

In this section, we define a *global B-spline basis* of the space $GS_2(\beta)$ of generalized quadratic splines.
Let $\omega_i = \dfrac{\theta_{i+1}h_{i+1}}{\theta_i h_i + \theta_{i+1}h_{i+1}}$ for $1 \le i \le n-1$.

Definitions
(i) For $1 \le i \le n-1$, let B_{2i} be the function whose support is the segment $[x_{i-1}, m_{i+1}]$ and whose lists of B-coefficients in the subintervals I_i and I_{i+1} are respectively

$$[0,0,1,\omega_i], \quad [\omega_i,0,0,0].$$

(ii) Similarly, let B_{2i+1} be the function whose support is the segment $[m_i, x_{i+1}]$, and whose lists of B-coefficients in the subintervals I_i and I_{i+1} are respectively

$$[0,0,0,1-\omega_i], \quad [1-\omega_i,1,0,0].$$

(iii) Moreover, there are 4 special B-splines $\{B_0, B_1, B_{2n}, B_{2n+1}\}$ at the end points of I, defined respectively by
supp(B_0)=$[x_0, m_0]$, its list of B-coefficients in I_1 being $[1,0,0,0]$.
supp(B_1)=$[x_0, x_1]$, its list of B-coefficients in I_1 being $[0,1,0,0]$.
supp(B_{2n})=$[x_{n-1}, x_n]$, its list of B-coefficients in I_n being $[0,0,1,0]$.
supp(B_{2n+1})=$[m_n, x_n]$, its list of B-coefficients in I_n being $[0,0,0,1]$.

It is easy to verify that

$$B_0'(x_0) = \frac{-1}{\eta_0 - x_0} = \frac{-1}{\theta_1 h_1}, \quad B_{2n}'(x_n) = \frac{-1}{x_n - \xi_n} = \frac{-1}{\theta_n h_n},$$

$$B_1'(x_0) = -B_0'(x_0), \quad B_{2n+1}'(x_n) = -B_{2n}'(x_n),$$

and for $1 \le i \le n-1$

$$B_{2i}'(x_i) = \frac{-1}{\eta_i - \xi_i} = -\frac{\omega_i}{\theta_{i+1}h_{i+1}} = -B_{2i+1}'(x_i).$$

Theorem 1. *The generalized quadratic B-splines $\{B_k, 0 \le k \le 2n+1\}$ form a basis of the space $GS_2(\beta)$. Moreover, they form a partition of unity, or a blending system, in this space.*

Proof: First, let us prove that the B-splines belong to the space $GS_2(\beta)$, i.e. that they are C^1 continuous at the points $\{x_1, x_2, \ldots, x_{n-1}\}$. By construction, they are already C^1 in each subinterval I_i. In addition, their derivatives satisfy the above relations, so they are continuous at the interior points of X.

The B-splines are linearly independent: assume that $g = \sum_{k=0}^{2n+1} \gamma_k B_k = 0$. Then, the B-coefficients of the restriction $g_i = \sum_{k=2i-2}^{2i+1} \gamma_k B_k$ of g to I_i are respectively

$$a_{i-1} = \omega_{i-1}\gamma_{2i-2} + (1-\omega_{i-1})\gamma_{2i-1}, \quad d_{i-1} = \gamma_{2i-1},$$

$$c_i = \gamma_{2i}, \quad a_i = \omega_i \gamma_{2i} + (1-\omega_i)\gamma_{2i+1}.$$

Therefore, since ω_{i-1} and $1-\omega_i$ are non zero, we get successively $\gamma_{2i-1} = \gamma_{2i} = 0$, and $\gamma_{2i-2} = \gamma_{2i+1} = 0$.
The B-splines generate the space $GS_2(\beta)$: it suffices to express the coefficients γ_k in function of the B-coefficients. The above equations give immediately

$$\gamma_{2i} = c_i, \quad \gamma_{2i+1} = d_i.$$

Finally, let us prove that $\sum B_k = 1$: it suffices to prove it on each subinterval I_i. As local B-spline bases are blending systems, one has to show that the sum of local B-coefficients of global B-splines is equal to 1. This property is easily deduced from the lists of B-coefficients of $B_{2i-2}, B_{2i-1}, B_{2i}, B_{2i+1}$ on the interval I_i which are respectively equal to

$$[\omega_{i-1}, 0, 0, 0], \;\; [1-\omega_{i-1}, 1, 0, 0], \;\; [0, 0, 1, \omega_i], \;\; [0, 0, 0, \omega_i]$$

□

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§3. Global and local control polygons. Monotonicity and convexity

It has been proved in [6] that the spaces $V(\beta_i)$ contain the space $\mathbb{P}_1$ of affine polynomials . As the list of local B-coefficients on I_i of the function $e_1(x) = x$ is $[x_{i-1}, \eta_{i-1}, \xi_i, x_i]$, the results of section 2 show that the representation of e_1 in the basis of generalized B-splines is given by

$$e_1 = \sum_{i=0}^{n+1} (\xi_i B_{2i} + \eta_i B_{2i+1}).$$

Therefore, we can define the global *spline control polygon* (abbr. SCP) of $g = \sum_{k=0}^{2n+1} \gamma_k B_k$ with vertices

$$\tilde{\gamma}_{2i} = (\xi_i, \gamma_{2i}), \quad \tilde{\gamma}_{2i+1} = (\eta_i, \gamma_{2i+1})$$

from which we easily deduce the vertices of the local control polygon (LCP)

$$\tilde{a}_{i-1} = \omega_{i-1}\tilde{\gamma}_{2i-2} + (1-\omega_{i-1})\tilde{\gamma}_{2i-1}, \quad \tilde{c}_{i-1} = \tilde{\gamma}_{2i-2},$$

$$\tilde{d}_i = \gamma_{2i+1}, \quad \tilde{a}_i = \omega_i \tilde{\gamma}_{2i} + (1-\omega_i)\tilde{\gamma}_{2i+1}.$$

Theorem 2. *A function $g \in GS_2(\beta)$ is monotonic (resp. convex) if and only if its SCP is monotonic (resp. convex), with the same sense of variation.*

Proof: ¿From theorem 6 of [10], we know that g is (e.g.) increasing on I_i if and only if its LCP is increasing, i.e. iff

$$a_{i-1} \le d_{i-1} \le c_i \le a_i.$$

Since we have

$$d_{i-1} - a_{i-1} = \omega_{i-1}(\gamma_{2i-1} - \gamma_{2i-2}), \quad c_i - d_{i-1} = \gamma_{2i} - \gamma_{2i-1},$$

$$a_i - c_i = (1-\omega_i)(\gamma_{2i+1} - \gamma_{2i}),$$

we see that the above inequalities are satisfied iff $\gamma_{2i-1} \le \gamma_{2i-1} \le \gamma_{2i-1}$, i.e. iff the global SCP is increasing.
Similarly, from theorem 8 of [10], we know that g is convex on I_i if and only if

$$\frac{d_{i-1} - a_{i-1}}{\eta_{i-1} - x_{i-1}} \le \frac{c_i - d_{i-1}}{\xi_i - \eta_{i-1}} \le \frac{a_i - c_i}{x_i - \xi_i}.$$

Using the equalities

$$\eta_{i-1} - \xi_{i-1} = \frac{\theta_i h_i}{\omega_{i-1}}, \;\; \xi_i - \eta_{i-1} = (1-2\theta_i)h_i, \;\; \eta_i - \xi_i = \frac{\theta_i h_i}{1-\omega_i}$$

we obtain successively the following identities

$$\frac{d_{i-1} - a_{i-1}}{\eta_{i-1} - x_{i-1}} = \frac{\omega_{i-1}(\gamma_{2i-1} - \gamma_{2i-2})}{\theta_i h_i} = \frac{\gamma_{2i-1} - \gamma_{2i-2}}{\eta_{i-1} - \xi_{i-1}},$$

$$\frac{c_i - d_{i-1}}{\xi_i - \eta_{i-1}} = \frac{\gamma_{2i} - \gamma_{2i-1}}{(1-2\theta_i)h_i} = \frac{\gamma_{2i} - \gamma_{2i-1}}{\xi_i - \eta_{i-1}},$$

$$\frac{a_i - c_i}{x_i - \xi_i} = \frac{(1-\omega_i)(\gamma_{2i+1} - \gamma_{2i})}{\theta_i h_i} = \frac{\gamma_{2i+1} - \gamma_{2i}}{\eta_i - \xi_i},$$

which show that the convexity of the global SCP is equivalent to that of the LCP in each subinterval, i.e. to the convexity of g. $\square$

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§4. Construction of monotone or/and convex interpolants

Algorithm 1: monotone interpolant

This algorithm describes the construction of an increasing interpolant $g \in GS_2(I, X, \beta)$ to arbitrary increasing Hermite data:

$$(y_i, p_i), \quad 0 \leq i \leq n,$$

assumed to satisfy the properties:

$$\Delta y_i > 0, \quad p_i > 0.$$

(There is a similar algorithm for decreasing data). The main point consists in choosing the sequence β in function of the data. ¿From theorem 2, we know that $g = \sum_k \gamma_k B_k$ is increasing if and only if the sequence (γ_k) of its S-coefficients is increasing. On one hand, we already know that

$$\gamma_{2i+1} - \gamma_{2i} = d_i - c_i = (\theta_{i+1} h_{i+1} + \theta_i h_i) p_i > 0.$$

On the other hand, using the notations:

$$\tau_i = \frac{\Delta y_{i-1}}{h_i}, \quad \mu_i = \frac{1}{2}(p_{i-1} + p_i),$$

we must have

$$\gamma_{2i} - \gamma_{2i-1} = c_i - d_{i-1} = h_i(\tau_i - 2\theta_i \mu_i) > 0,$$

which implies the following condition on the parameter $\theta_i = \frac{1}{2}\beta_i\beta_i - 1$:

$$0 < \theta_i < \bar{\theta}_i = \frac{1}{2}\frac{\tau_i}{\mu_i}.$$

Note that this condition is equivalent to $g'(m_i) > 0$.

There appear two cases and we obtain the following

Theorem 3. *(i) if $\mu_i \leq 2\tau_i$, then we can choose $\theta_i = \frac{1}{4}$: in that case, the local interpolant in the subinterval I_i is an ordinary quadratic spline ($\beta_i = -1$).*

(ii) if $\mu_i > 2\tau_i$, then we have to choose $\theta_i \leq \bar{\theta}_i = \frac{1}{2}\frac{\tau_i}{\mu_i} < \frac{1}{4}$: in that case, the local interpolant in the subinterval I_i is a generalized quadratic spline associated with the parameter $\beta_i = \frac{-2\theta_i}{1-2\theta_i}$.

Algorithm 2: convex interpolant

This algorithm describes the construction of a convex interpolant $g \in GS_2(I, X, \beta)$ to arbitrary convex Hermite data:

$$(y_i, p_i), \ \ 0 \leq i \leq n,$$

assumed to satisfy the properties

$$p_{i-1} < \tau_i < p_i$$

on each subinterval I_i. (There is a similar algorithm for concave data). Using the results given in the proof of theorem 2, we get the convexity conditions on I_i for the parameter $\theta_i = \frac{1}{2}\beta_i\beta_i - 1$:

$$p_{i-1} \leq \frac{\tau_i - 2\theta_i\mu_i}{(1-2\theta_i)} \leq p_i$$

which can also be written

$$(1-\theta_i)p_{i-1} + \theta_i p_i \leq \tau_i \leq (1-\theta_i)p_i + \theta_i p_{i-1}$$

As for algorithm 1, there appear two cases:

Theorem 4.

(i) if τ_i satisfies the two inequalities $\frac{1}{4}(3p_{i-1} + p_i) \leq \tau_i \leq \frac{1}{4}(p_{i-1} + 3p_i)$, then we chose $\theta_i = \frac{1}{4}$: in that case, the local interpolant is an ordinary quadratic spline.

(ii) else the local interpolant is a generalized quadratic spline ($\theta_i < \frac{1}{4}$).

a) either $p_{i-1} < \tau_i < \frac{1}{4}(3p_{i-1} + p_i)$: then we have to choose $\theta_i \leq \bar{\theta}_i = \frac{\tau_i - p_{i-1}}{p_i - p_{i-1}}$.

b) or $\frac{1}{4}(p_{i-1} + 3p_i) < \tau_i < p_i$: then we have to choose $\theta_i \leq \bar{\theta}_i = \frac{p_i - \tau_i}{p_i - p_{i-1}}$.

In both cases, the GQS belongs to $V(\beta_i)$, with $\beta_i = \frac{-2\theta_i}{1-2\theta_i}$.

Algorithm 3: monotone and convex interpolant

For sake of simplicity, we assume that the data are increasing and convex

$$0 < p_{i-1} < \tau_i < p_i$$

on each subinterval I_i. (There are similar algorithms for the three other cases). By putting together conditions of theorems 3 and 4, we get the same algorithm as in the preceding case.

Theorem 5. *(i) if* $\frac{1}{4}(3p_{i-1}+p_i) \le \tau_i \le \frac{1}{4}(p_{i-1}+3p_i)$, *then we can choose* $\theta_i = \frac{1}{4}$. *The local interpolant is an ordinary quadratic spline.*

(ii) else the local interpolant is a generalized quadratic spline.

a) either $\tau_i < \frac{1}{4}(3p_{i-1}+p_i)$, *then we chose* $\theta_i \le \bar{\theta}_i = \frac{\tau_i - p_{i-1}}{p_i - p_{i-1}}$.

b) or $\tau_i > \frac{1}{4}(p_{i-1}+3p_i)$, *then we chose* $\theta_i \le \bar{\theta}_i = \frac{p_i - \tau_i}{p_i - p_{i-1}}$.

In both cases, the GQS belongs to $V(\beta_i)$, with $\beta_i = \frac{-2\theta_i}{1-2\theta_i}$.
Remark: these are only rough algorithms: in practice, one has to smooth a little bit the above conditions and also to treat the cases when there appear equalities in the conditions. This will be done in a further more complete paper illustrated with numerical examples.

algorithm ***********************************

§5. Refinement equations and global corner-cutting algorithm

In this section, we consider the subpartition $\bar{X} = X \cup \{m_i, 1 \le i \le n\}$ dividing I into $2n$ subintervals. The space $GS_2(I, X, \beta)$ is a subspace of dimension $2n+2$ of the new space $GS_2(I, \bar{X}, \bar{\beta})$ of dimension $4n+2$. Here $\bar{\beta}$ denotes the sequence of parameters deduced from β by taking twice the same parameter β_i, once for the left subinterval $I'_i = [x_{i-1}, m_i]$ and once for the right subinterval $I''_i = [m_i, x_i]$ of I_i:

$$\bar{\beta} = (\beta_1, \beta_1, \beta_2, \beta_2, \ldots, \beta_n, \beta_n).$$

The finer B-splines of $GS_2(I, \bar{X}, \beta)$ are denoted

$$\{\bar{B}_l,\ 0 \le l \le 4n+1\}.$$

Thanks to the local corner-cutting algorithm (abbr. CCA, see [8], section), one can compute the local B-coefficients in the subintervals I'_i and I''_i of the coarser B-splines $\{B_k, 0 \le k \le n+1\}$ in function of their previous B-coefficients in I_i, and we get the following refinement equations.
Let us recall that $\omega_i = \dfrac{\theta_{i+1}h_{i+1}}{\theta_i h_i + \theta_{i+1}h_{i+1}}$, for $1 \le i \le n-1$, $\omega_0 = 1 - \omega_n = 1$.

Theorem 6. *(i) For all* $1 \le i \le n-1$, *one has the two following refinement equations:*

$$B_{2i} = (\frac{1}{2} + \frac{1}{4}\beta_i)\bar{B}_{4i-2} + (\frac{1}{2} - \frac{1}{4}\beta_i)\bar{B}_{4i-1} + \frac{1}{2}(1+\omega_i)\bar{B}_{4i} + \frac{1}{2}\omega_i\bar{B}_{4i+1},$$

$$B_{2i+1} = \frac{1}{2}(1-\omega_i)\bar{B}_{4i} + \frac{1}{2}(2-\omega_i)\bar{B}_{4i+1} + (\frac{1}{2} - \frac{1}{4}\beta_i)\bar{B}_{4i+2} + (\frac{1}{2} + \frac{1}{4}\beta_i)\bar{B}_{4i+3}.$$

(ii) For the B–splines at the endpoints, one has respectively:

$$B_0 = \bar{B}_0 + \frac{1}{2}\bar{B}_1,\ \ B_1 = \frac{1}{2}\bar{B}_1 + (\frac{1}{2} - \frac{1}{4}\beta_1)\bar{B}_2 + (\frac{1}{2} + \frac{1}{4}\beta_1)\bar{B}_3,$$

$$B_{2n} = \frac{1}{2}\bar{B}_{4n}+(\frac{1}{2}-\frac{1}{4}\beta_n)\bar{B}_{4n-1}+(\frac{1}{2}+\frac{1}{4}\beta_n)\bar{B}_{4n-2},\ \ B_{2n+1} = \frac{1}{2}\bar{B}_{4n}+\bar{B}_{4n+1}.$$

Proof: We only give the proof for B_{2i}, the others being similar. Let

$$B_{2i} = \mu_{4i-2}\bar{B}_{4i-2} + \mu_{4i-1}\bar{B}_{4i-1} + \mu_{4i}\bar{B}_{4i} + \mu_{4i+1}\bar{B}_{4i+1}.$$

By application of the CCA, starting from the B-coefficients $[0, 0, 1, \omega_i]$ of B_{2i} on the interval I_i, we deduce the B-coefficients of B_{2i}, respectively on the subintervals I'_i and I''_i:

$$[0, 0, \frac{1}{2} + \frac{1}{4}\beta_i, \frac{1}{2}] \ \ \textit{and} \ \ [\frac{1}{2}, \frac{1}{2} - \frac{1}{4}\beta_i, \frac{1}{2} + \frac{1}{2}\omega_i, \omega_i].$$

Similarly, from the B-coefficients $[\omega_i, 0, 0, 0]$ of B_{2i} on I_{i+1}, we deduce the B-coefficients of B_{2i} on the subintervals I'_{i+1} and I''_{i+1}:

$$[\omega_i, \frac{1}{2}\omega_i, 0, 0] \ \ \textit{and} \ \ [0, 0, 0, 0]$$

On the other hand, the B-coefficients of the finer B-splines are respectively

1) on the subintervals I'_i and I''_i

$$\textit{for} \ \ \bar{B}_{4i-2}: \ \ [0, 0, 1, \frac{1}{2}], \ \ [\frac{1}{2}, 0, 0, 0]$$

$$\textit{for} \ \ \bar{B}_{4i-1}: \ \ [0, 0, 0, \frac{1}{2}], \ \ [\frac{1}{2}, 1, 0, 0]$$

2) on the subintervals I''_i and I'_{i+1}:

$$\textit{for} \ \ \bar{B}_{4i}: \ \ [0, 0, 1, \omega_i], \ \ [\omega_i, 0, 0, 0]$$

$$\textit{for} \ \ \bar{B}_{4i+1}: \ \ [0, 0, 0, 1-\omega_i], \ \ [1-\omega_i, 1, 0, 0]$$

Therefore, the B-coefficients of B_{2i} as linear combination of the four finer B-splines on the three subintervals I'_i, I''_i and I'_{i+1} are respectively equal to

$$[0, 0, \mu_{4i-2}, \frac{1}{2}(\mu_{4i-2}+\mu_{4i-1})], \ \ [\frac{1}{2}(\mu_{4i-2}+\mu_{4i-1}), \mu_{4i-1}, \mu_{4i}, \omega_i\mu_{4i}+(1-\omega_i)\mu_{4i+1}]$$

$$[\omega_i\mu_{4i} + (1-\omega_i)\mu_{4i+1}, \mu_{4i+1}, 0, 0].$$

By identifying these B-coefficients with those of B_{2i}, one obtains:

$$\mu_{4i-2} = \frac{1}{2} + \frac{1}{4}\beta_i, \ \mu_{4i-1} = \frac{1}{2} - \frac{1}{4}\beta_i, \ \mu_{4i} = \frac{1}{2}(1+\omega_i), \ \mu_{4i+1} = \frac{1}{2}\omega_i$$

□

An immediate consequence of the previous theorem is the following *global corner-cutting algorithm*:

Theorem 7. *Given the two expansions of $S \in GS_2(I, X, \beta)$ with respect to the coarse and fine B-splines bases:*

$$S = \sum_{k=0}^{2n+1} \gamma_k B_k = \sum_{l=0}^{4n+1} \delta_l \bar{B}_l,$$

then, the new B-coefficients have the following expressions in terms of the former B-coefficients:

$$\delta_{4i-2} = (\frac{1}{2} - \frac{1}{4}\beta_i)\gamma_{2i-1} + (\frac{1}{2} + \frac{1}{4}\beta_i)\gamma_{2i},$$

$$\delta_{4i-1} = (\frac{1}{2} + \frac{1}{4}\beta_i)\gamma_{2i-1} + (\frac{1}{2} - \frac{1}{4}\beta_i)\gamma_{2i},$$

$$\delta_{4i} = \frac{1}{2}(1+\omega_i)\gamma_{2i} + \frac{1}{2}(1-\omega_i)\gamma_{2i+1},$$

$$\delta_{4i+1} = \frac{1}{2}\omega_i\gamma_{2i} + \frac{1}{2}(2-\omega_i)\gamma_{2i+1}.$$

Proof: The proof simply consists in comparing the coefficients in the two expressions after substituting in the first expression the coarser B-splines B_k by their expansions as linear combinations of the finer B-splines $\bar{B}_l$, given in theorem 3. $\square$

Theorem 8. *The sequence of SCPs associated with a given GQS, obtained by successive applications of the global CCA, converges uniformly to the GQS.*

Proof: Let $S(x) = \sum_{k=0}^{2n+1} \gamma_k B_k$ be the equation of a GQS and let $P_0 = \sum_{k=0}^{2n+1} \gamma_k \phi_k$ be the equation of its initial SCP. We denote by ϕ_{2i} the hat function with support $[\eta_{i-1}, \eta_i]$ satisfying $\phi_{2i}(\xi_i) = 1$, and by ϕ_{2i+1} the hat function with support $[\xi_i, \xi_{i+1}]$ satisfying $\phi_{2i}(\eta_i) = 1$.

For $x \in [\eta_{i-1}, \xi_i]$, (resp. $x \in [\xi_i, \eta_i]$), we have

$$|S(x) - P_0(x)| \le \max\{|S(x) - \gamma_{2i-1}|, |S(x) - \gamma_{2i}|\}$$

$$(resp. |S(x) - P_0(x)| \le \max\{|S(x) - \gamma_{2i}|, |S(x) - \gamma_{2i+1}|\}).$$

Moreover, for $x \in [\eta_{i-1}, \xi_i]$, we have

$$|S(x) - \gamma_{2i}| \le \sum_{k=2i-2}^{2i+1} |\gamma_k - \gamma_{2i}| B_k(x) \le \max\{|\gamma_k - \gamma_{2i}|, 2i-2 \le k \le 2i+1\}$$

As we observe that

$$|\gamma_k - \gamma_{2i}| \le 2\max\{|\gamma_{k+1} - \gamma_k|, 2i-2 \le k \le 2i+1\},$$

we are led to define

$$\Delta_0 = max\{|\gamma_{k+1} - \gamma_k|, 0 \le k \le 2n\},$$

and we finally obtain

$$\|S - P_0\|_\infty \le 2\Delta_0.$$

Now, we do the same for the next SCP P_1 deduced from P_0 by one application of the global CCA. Here, we define

$$\Delta_1 = \max\{|\delta_{l+1} - \delta_l|, 0 \le l \le 4n+1\}.$$

We have successively the following majorations

$$|\delta_{4i-1} - \delta_{4i-2}| = |\frac{1}{2}\beta_i\ (\gamma_{2i} - \gamma_{2i-1})| \le \frac{1}{2}|\gamma_{2i} - \gamma_{2i-1}|$$

$$|\delta_{4I+1} - \delta_{4i}| = \frac{1}{2}|\gamma_{2i+1} - \gamma_{2i}|$$

$$\begin{aligned}|\delta_{4i} - \delta_{4i-1}| &= |\frac{1}{2}(1-\omega_i)(\gamma_{2i+1} - \gamma_{2i}) + (\frac{1}{2} + \frac{1}{4}\beta_i)(\gamma_{2i} - \gamma_{2i-1}))| \\ &\le \frac{1}{2}\max\{|\gamma_{2i+1} - \gamma_{2i}|, |\gamma_{2i} - \gamma_{2i-1}|\}\end{aligned}$$

Therefore we obtain

$$\Delta_1 \le \frac{1}{2}\Delta_0.$$

Denoting by Δ_m the maximum distance between two consecutive vertices of the SCP P_m obtained after m applications of the global CCA, we get

$$\Delta_m \le \frac{1}{2^m}\Delta_0,$$

which proves the uniform convergence to the GQS g of the sequence (P_n) of its SCPs. □

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§6. Quasi-interpolant and Lagrange interpolant

In this section, we define two kinds of approximation operators: the first is a quasi-interpolant with good shape-preserving properties while the second is a Lagrange interpolant which is uniformly bounded independently of the partition. Let us begin with the *quasi-interpolant* Q defined by

$$Qf = \sum_{i=0}^{n+1}[f(\xi_i)B_{2i} + f(\eta_i)B_{2i+1}]$$

Theorem 9. *Q is exact on $\mathbb{P}_1$, $\|Q\|_\infty = 1$, and Q preserves the monotonicity and the convexity of f.*

Proof: Since $e_0 = \sum_{i=0}^{n+1}[B_{2i} + B_{2i+1}]$ and $e_1 = \sum_{i=0}^{n+1}[\xi_i B_{2i} + \eta_i B_{2i+1}]$, we obtain immediately the property that Q is exact on $\mathbb{P}_1$. Moreover, $Qe_0 = e_0$ and $|Qf|_\infty \leq |f|_\infty$ imply $\|Q\|_\infty = 1$.
If f is increasing (resp. convex), the global B-polygon of Qf is also increasing (resp. convex), and the result follows by a direct application of theorem 2. □

Now, let us study a *Lagrange interpolation operator* L in the space $GS_2(\beta)$. Let m'_i (resp. m''_i) be the midpoint of $I'_i = [x_{i-1}, m_i]$ (resp. of $I''_i = [m_i, x_i]$. The following theorem is an extension to GQS of a result previously given by Kammerer et al. [4] and by Marsden [5] for ordinary quadratic splines.

Theorem 10. *(i) Given a function f defined on $I = [a,b]$, there exists a unique generalized quadratic spline $Lf \in GS_2(I, X, \beta)$ satisfying the following interpolation properties:*

$$Lf(a) = f(a),\ \ Lf(b) = f(b)\ \ and\ \ for\ \ 1 \leq i \leq n,$$

$$Lf(m'_i) = f(m'_i)\ \ and\ \ Lf(m''_i) = f(m''_i).$$

(ii) The B-coefficients of $Lf = \sum_{k=0}^{2n+1} \gamma_k B_k$ are solutions of the tridiagonal system of $2n$ linear equations $(1 \leq i \leq n)$:

$$\omega_{i-1}\gamma_{2i-2} + (3 - \omega_{i-1} - \frac{1}{2}\beta_i)\gamma_{2i-1} + (1 + \frac{1}{2}\beta_i)\gamma_{2i} = 4f(m'_i)$$

$$(1 + \frac{1}{2}\beta_i)\gamma_{2i-1} + (2 + \omega_i - \frac{1}{2}\beta_i)\gamma_{2i} + (1 - \omega_i)\gamma_{2i+1} = 4f(m''_i)$$

with $\gamma_0 = f(a)$, $\gamma_{2n+1} = f(b)$.

Proof: On the interval I_i, we are able to compute the values of $Lf = \sum_{k=2i-2}^{2i+1} \gamma_k B_k$ at the points m'_i and m''_i by applying the CCA to B-splines. Without going into details, it is straightforward to obtain the coefficients of the two equations given in the theorem. The tridiagonal matrix of the linear system is strictly diagonally dominant since the two corresponding inequalities

$$3 - \omega_{i-1} - \frac{1}{2}\beta_i > \omega_{i-1} + 1 + \frac{1}{2}\beta_i\ \ and\ \ 2 + \omega_i - \frac{1}{2}\beta_i > 1 + \frac{1}{2}\beta_i + 1 - \omega_i$$

are respectively equivalent to the following

$$2(1 - \omega_i) - \beta_i > 0\ \ and\ \ 2\omega_i - \beta_i > 0$$

which are obviously satisfied since $\beta_i < 0$ and $0 < \omega_i < 1$. $\square$

Theorem 11. *$\|L\|_\infty$ is uniformly bounded for all partitions of I. More specifically, setting $\bar{\beta} = max\{\beta_i, 1 \le i \le n\}$, we obtain*

$$\|L\|_\infty \le \frac{4(3\bar{\beta}-1)}{\bar{\beta}(5-3\bar{\beta})}.$$

Proof: For this purpose, we start from the expression of $Lf = \sum_{r=0}^{3} a_{i,r} b_r$ in the local B-spline basis of each subinterval I_i. Then, by the CCA, we compute the values of the b_r at the two points m'_i and m''_i and we get the two equations:

$$4Lf(m'_i) = a_{i,0} + (2 - \frac{1}{2}\beta_i)a_{i,1} + (1 + \frac{1}{2}\beta_i)a_{i,2} = 4f(m'_i)$$

$$4Lf(m''_i) = (1 + \frac{1}{2}\beta_i)a_{i,1} + (2 - \frac{1}{2}\beta_i)a_{i,2} + a_{i,3} = 4f(m'_i)$$

We have to add to these equations the C^1 continuity conditions of Lf at interior points x_i, which can be written:

$$a_{i-1,3} = a_{i,0} \;\; and \;\; \lambda_{i-1}(a_{i-1,3} - ai - 1, 2) = \lambda_i(a_{i,1} - a_{i,0})$$

where $\lambda_i = 2\left(\frac{\beta_i - 1}{\beta_i}\right) = \frac{1}{\theta_i}$ for all $1 \le i \le n$.
¿From these conditions, we deduce the expressions

$$a_{i-1,3} = a_{i,0} = \frac{\lambda_{i-1}a_{i-1,2} + \lambda_i a_{i,1}}{\lambda_{i-1} + \lambda_i}$$

that we substitute in the two equations above. Taking as unknowns $a_{2i-1} = a_{i,1}$ and $a_{2i} = a_{i,2}$, we then obtain the following tridiagonal system of equations

$$\frac{\lambda_{i-1}}{\lambda_{i-1} + \lambda_i}a_{2i-2} + \left[\frac{\lambda_i}{\lambda_{i-1} + \lambda_i} + 2 - \frac{1}{2}\beta_i\right] a_{2i-1} + (1 + \frac{1}{2}\beta_i)a_{2i} = 4f(m'_i)$$

$$(1 + \frac{1}{2}\beta_i)a_{2i-1} + \left[\frac{\lambda_i}{\lambda_i + \lambda_{i+1}} + 2 - \frac{1}{2}\beta_i\right] a_{2i} + \frac{\lambda_{i+1}}{\lambda_i + \lambda_{i+1}}a_{2i+1} = 4f(m''_i)$$

The tridiagonal matrix is strictly diagonally dominant since the two conditions

$$\frac{\lambda_i}{\lambda_{i-1} + \lambda_i} + 2 - \frac{1}{2}\beta_i > \frac{\lambda_{i-1}}{\lambda_{i-1} + \lambda_i} + 1 + \frac{1}{2}\beta_i,$$

$$\frac{\lambda_i}{\lambda_i + \lambda_{i+1}} + 2 - \frac{1}{2}\beta_i > 1 + \frac{1}{2}\beta_i + \frac{\lambda_{i+1}}{\lambda_i + \lambda_{i+1}},$$

can also be written

$$-\beta_i > \frac{-2\lambda_i}{\lambda_{i-1}+\lambda_i} \;\; and \;\; -\beta_i > \frac{-2\lambda_i}{\lambda_{i+1}+\lambda_i},$$

and they are obviously satisfied since $\beta_i < 0$ and $\lambda_i > 0$ for all i.
Let $|a|_\infty = max\{|a_k|, 0 \le k \le 2n+1\}$. ¿From the above tridiagonal system, we deduce respectively the following inequalities

$$\left[\frac{\lambda_i}{\lambda_{i-1}+\lambda_i} + 2 - \frac{1}{2}\beta_i\right]|a_{2i-1}| \le 4|f(m_i')| + \frac{\lambda_{i-1}}{\lambda_{i-1}+\lambda_i}|a_{2i-2}| + (1+\frac{1}{2}\beta_i)|a_{2i}|,$$

$$\left[\frac{\lambda_i}{\lambda_i+\lambda_{i+1}} + 2 - \frac{1}{2}\beta_i\right]|a_{2i}| \le 4|f(m_i'')| + (1+\frac{1}{2}\beta_i)|a_{2i-1}| + \frac{\lambda_{i+1}}{\lambda_i+\lambda_{i+1}}|a_{2i+1}|,$$

and we obtain for all i:

$$\left[\frac{2\lambda_i}{\lambda_{i-1}+\lambda_i} - \beta_i\right]|a|_\infty \le 4\|f\|_\infty \;\; and \;\; \left[\frac{2\lambda_i}{\lambda_{i+1}+\lambda_i} - \beta_i\right]|a|_\infty \le 4\|f\|_\infty.$$

Defining $\bar\lambda = max\{\lambda_i, 1 \le i \le n\}$ and $\bar\beta = max\{\beta_i, 1 \le i \le n\}$, then we have $4 \le \lambda_i \le \bar\lambda = 2\frac{\bar\beta-1}{\bar\beta}$, hence

$$\frac{2\lambda_i}{\lambda_{i-1}+\lambda_i} - \beta_i \ge \frac{8}{\lambda_{i-1}+4} - \bar\beta \ge \frac{8}{\bar\lambda+4} - \bar\beta = \frac{\bar\beta(5-3\bar\beta)}{(3\bar\beta-1)}$$

Therefore, we finally obtain

$$|a|_\infty \le \frac{4(3\bar\beta-1)}{\bar\beta(5-3\bar\beta)}\|f\|_\infty.$$

As the graph of Lf lies in the convex hull of its local control polygons, it is not difficult to see that we have also

$$\|Lf\|_\infty \le \frac{4(3\bar\beta-1)}{\bar\beta(5-3\bar\beta)}\|f\|_\infty.$$

In other words, we obtain the following upper bound which is independent of the given partition, but only depends on the sequence β:

$$\|L\|_\infty \le \frac{4(3\bar\beta-1)}{\bar\beta(5-3\bar\beta)}.$$

□

Remarks: 1. In the case of classical quadratic splines, we have $\bar\beta = -1$, whence $\|L\|_\infty \le 2$, which is the sharp upper bound given in [4] and [5]. This suggests that the upper bound given in theorem 11 may also be sharp.

2. As for error bounds, we know, from a classical result in approximation theory (see [3], theorem), that for $P = Q$ or L

$$\|f - Pf\|_\infty \le (1 + \|P\|_\infty) d_\infty(f, GS_2(\beta)).$$

For $f \in C^2(I)$, we already know (see [3]) that $d_\infty(f, GS_2(\beta)) = O(h^2)$, with $h = \max h_i$. Therefore, we see that both operators have an approximation order equal to 2.

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